

On weak Fano manifolds with small contractions obtained by blow-ups of a product of projective spaces

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Abstract

We consider weak Fano manifolds with small contractions obtained by blowing up successively curves and subvarieties of codimension 2 in products of projective spaces. We give a classification result for a special case. In the process of proof, we describe explicitly the structure of nef cones and compute the self intersection numbers of anti-canonical divisors for such weak Fano manifolds.

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1 Introduction

A smooth projective variety is called *Fano manifold* if its anti-canonical divisor is ample. The classification is known up to dimension 3. However, in dimension greater than or equal to 4, there exist only partial classification results (see [2] for a recent progress).

It is essential to investigate Fano manifolds in terms of the theory of extremal contractions (see [8],[10]). Recall that a *small contraction* is a birational morphism whose exceptional locus has codimension greater than or equal to 2, and it does not appear as extremal contraction for smooth 3-folds. Hence, in dimension greater than or equal to 4, it is interesting to give examples of Fano manifolds having small contractions.

We can construct a smooth projective variety with a small contraction by means of successive blow-ups (see [5]): Let Y be a smooth projective variety of dimension greater than or equal to 4. Let C be a smooth curve on Y and S a smooth subvariety of Y with $\text{codim}_Y S = 2$. Assume that C and S intersect transversally at points. Let $\pi : X \rightarrow Y$ be the blow-up along C and let S' be the strict transform of S by π . Let $\beta : \tilde{X} \rightarrow X$ be the blow-up along S' . Then \tilde{X} has a small contraction (see Section 2 for details). We consider the following:

Problem. Classify the triples (Y, C, S) such that \tilde{X} is a Fano manifold.

The purpose of this paper is to give a classification result in a special case for the problem expanded to the case where \tilde{X} is a *weak Fano manifold*, i.e. a smooth projective variety with nef and big anti-canonical divisor.

Throughout the paper, we work over the field of complex numbers.

Theorem 1. *Let $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$ with $n \geq 3$. Let C be a fiber of the projection $Y \rightarrow \mathbb{P}^{n-1}$ and let S be a complete intersection of two divisors of bidegrees (a, b) and $(1, 1)$. Assume that S is smooth and irreducible. Assume also that S and C intersect transversally at one point. Let $\pi : X \rightarrow Y$ be the blow-up along C and let $\beta : \tilde{X} \rightarrow X$ be the blow-up along the strict transform of S by π . Then \tilde{X} is a weak Fano manifold if and only if $n \geq 3$ and*

$$(a, b) = (0, 1), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1) \text{ or } (3, 2).$$

Moreover, \tilde{X} is a Fano manifold if and only if $n \geq 4$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (2, 0) \text{ or } (2, 1).$$

Remarks: (1) The case $Y = \mathbb{P}^n$ seems more complicated (see Section 6).

(2) The assumption on C is not so restrictive. Indeed, if C is not a fiber of the projection $p : Y = \mathbb{P}^{n-1} \times \mathbb{P}^1 \rightarrow \mathbb{P}^{n-1}$, there exists a fiber Γ of p such that $C \cap \Gamma \neq \emptyset$. Then we have $-K_{\tilde{X}} \cdot \tilde{\Gamma} = 4 - n$, $\tilde{\Gamma}$ being the strict transform of Γ by $\pi \circ \beta$. Hence, $-K_{\tilde{X}}$ is not nef for $n \geq 5$.

(3) Let $q : Y = \mathbb{P}^{n-1} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection. Put $y_0 := C \cap S$. Since we assume S to be irreducible, $a = 0$ implies $b = 1$ and S is a hyperplane in the fiber $q^{-1}(q(y_0)) \simeq \mathbb{P}^{n-1}$. If $a \geq 1$, then $q|_S : S \rightarrow \mathbb{P}^1$ is surjective. The

assumption that S is contained in a divisor of bidegree $(1,1)$ is natural (at least for the case where \tilde{X} is a Fano manifold): Consider the open set

$$T := \{t \in \mathbb{P}^1 \mid t \neq q(y_0) \text{ and } S \cap q^{-1}(t) \text{ is smooth} \}.$$

If \tilde{X} is a Fano manifold, so is $\tilde{X}_t := (q \circ \pi \circ \beta)^{-1}(t)$ for $t \in T$. Note that $(\pi \circ \beta)|_{\tilde{X}_t} : \tilde{X}_t \rightarrow q^{-1}(t) \simeq \mathbb{P}^{n-1}$ is the blow-up whose center consists of the point $C \cap q^{-1}(t)$ and the subvariety $S_t := S \cap q^{-1}(t)$. According to [1], there exist a hypersurface $U_t \subset q^{-1}(t) \simeq \mathbb{P}^{n-1}$ of degree a ($1 \leq a \leq n$) and a hyperplane $V_t \subset q^{-1}(t) \simeq \mathbb{P}^{n-1}$ such that S_t is complete intersection of U_t and V_t . Let V be the closure of the union $\bigcup_{t \in T} V_t$. Then, V contains S and V has bidegree $(1, c)$ for some $c \geq 0$, and our theorem covers the case $c = 1$.

The present paper is organized as follows: In Section 2, we explain how to obtain a small contraction by means of blow-ups. We also fix notations which will be used constantly throughout the paper. Section 3 is devoted to determine the structure of the nef cones of \tilde{X} for $(a, b) = (1, 0)$ and for any (a, b) such that $a \geq 1$ and $b \geq 0$. Recently, the explicite descriptions of nef cones are of great importance in the study of Mori dream spaces (see [9]). Hence, this section is of independent interest. In Section 4, we compute $(-K_{\tilde{X}})^n$ and express it as a rational function depending on (n, a, b) . We will give a sufficient condition for $(-K_{\tilde{X}})^n$ to be strictly positive. Since the self intersection number of the anti-canonical divisor is an important invariant for (weak) Fano manifolds, we believe that this section is also of independent interest. In Section 5, we prove Theorem 1 using Propositions shown in Sections 3 and 4. Section 6 is a supplement in which we give several examples for the case $Y \neq \mathbb{P}^{n-1} \times \mathbb{P}^1$.

Notation. Let $(x_0 : x_1 : \cdots : x_{n-1})$ and $(s : t)$ are homogeneous coordinates of \mathbb{P}^{n-1} and \mathbb{P}^1 respectively. Recall that a divisor D on the product $\mathbb{P}^{n-1} \times \mathbb{P}^1$ is said to have bidegree (a, b) if D is defined by a polynomial

$$\sum c_{i_0, i_1, \dots, i_{n-1}, j, k} x_0^{i_0} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} s^j t^k \quad (c_{i_0, \dots, i_{n-1}, j, k} \in \mathbb{C})$$

such that $i_0 + \cdots + i_{n-1} = a$, $j + k = b$. It is equivalent to say that D is a member of the linear system $|\mathcal{O}_{\mathbb{P}^{n-1} \times \mathbb{P}^1}(a, b)|$.

For a projective variety X , we denote by $N^1(X)$ (resp. $N_1(X)$) the set of the numerical classes of divisors (resp. 1-cycles) with real coefficients.

It is known that this is a finite dimensional vector space (see [6]), and its dimension denoted by $\rho(X)$ is called the *Picard number* of the variety X . The numerical equivalence class of a divisor D (resp. a 1-cycle C) is denoted by $[D]$ (resp. $[C]$). We see that $N^1(X)$ and $N_1(X)$ are dual to each other via the bilinear form $N^1(X) \times N_1(X) \rightarrow \mathbb{R}$ defined by the intersection number : $([D], [C]) \mapsto D \cdot C$.

The *nef cone* $Nef(X)$ and the *cone of curves* $NE(X)$ are defined by

$$\begin{aligned} Nef(X) &:= \{[D] \in N^1(X) \mid D \text{ is a nef divisor}\}, \\ NE(X) &:= \left\{ \sum a_i [C_i] \in N_1(X) \mid C_i \text{ is an irreducible curve on } X, a_i \geq 0 \right\}. \end{aligned}$$

The closure of $NE(X)$ in $N_1(X)$ is denoted by $\overline{NE}(X)$. The important fact is that the two cones $Nef(X)$ and $\overline{NE}(X)$ are dual to each other (see [7] Proposition 1.4.28).

Let Γ be a 1-cycle on a projective variety Y and let V be a subvariety of Y . For a divisor D on V , we denote by $(D \cdot \Gamma)_V$ the intersection number taken in V . Given a birational morphism $\alpha : X \rightarrow Y$, the strict transform of a subvariety $M \subset Y$ will be denoted by $\alpha_*^{-1}M$.

2 Construction of a small contraction

We follow Example (2.6) in [5]. Let Y be a smooth projective variety of dimension $n \geq 3$. Let $C \subset Y$ be a smooth curve and let $S \subset Y$ be a smooth subvariety of codimension 2. Assume that C and S intersect transversally at one point. Put $y_0 := S \cap C$. Let $\pi : X \rightarrow Y$ be the blow-up along C with the exceptional divisor E . Note that $\pi|_E : E \rightarrow C$ is a \mathbb{P}^{n-2} -bundle. Put $E_0 := \pi^{-1}(y_0)$. Let $\beta : \tilde{X} \rightarrow X$ be the blow-up along $S' := \pi_*^{-1}S$ with the exceptional divisor F . Let f be a fiber of the \mathbb{P}^1 -bundle $\beta|_F : F \rightarrow S'$. We put $\tilde{E} := \beta^{-1}(E)$ and $\tilde{E}_0 := \beta_*^{-1}E_0$. Note that \tilde{E}_0 is isomorphic to \mathbb{P}^{n-2} .

Lemma 1. *There exists a birational morphism $\varphi : \tilde{X} \rightarrow X_0$, X_0 being a projective variety, such that $\varphi(\tilde{E}_0)$ is a point for $n \geq 4$. The same holds for $n = 3$, if we assume $-K_{\tilde{X}}$ is nef and big.*

Proof. (See also [3] Chapter 6.) Let \tilde{e}_0 be a line in $\tilde{E}_0 \simeq \mathbb{P}^{n-2}$. We show that $\mathbb{R}^+[\tilde{e}_0]$ is extremal in the cone $\overline{NE}(\tilde{X})$. Assume that there exist irreducible curves $A, B \subset \tilde{X}$ such that $\tilde{e}_0 \equiv A + B$. Let D be an ample divisor

on Y and put $\widetilde{D} := (\pi \circ \beta)^* D$. Since $\widetilde{D} \cdot \widetilde{e}_0 = 0$, we have $\widetilde{D} \cdot A = \widetilde{D} \cdot B = 0$, which implies that A and B are contracted by $\pi \circ \beta$. Assume $A \not\subset \widetilde{E}$. Then there exists $s \in S \setminus \{y_0\}$ such that $A = (\pi \circ \beta)^{-1}(s)$. Since $(\pi \circ \beta)(B)$ is a point, B is one of the following types:

1. $B = (\pi \circ \beta)^{-1}(t)$ ($t \in S \setminus \{y_0\}$)
2. $B \subset (\pi \circ \beta)^{-1}(c)$ ($c \in C \setminus \{y_0\}$)
3. $B \subset (\pi \circ \beta)^{-1}(y_0)$

In case 1, we have $\widetilde{e}_0 \equiv A + B \equiv f + f = 2f$, a contradiction. In case 2, we have $F \cdot A + F \cdot B = -1 + 0 = -1$, while $F \cdot (A + B) = F \cdot \widetilde{e}_0 = 1$, a contradiction. In case 3, if we put $G := F \cap \widetilde{E}$ then we have $(\pi \circ \beta)^{-1}(y_0) = \widetilde{E}_0 \cup G$. Assume $B \subset G$. Put $G_0 := F \cap \widetilde{E}_0$. Note that $N_{G_0/G} \simeq \mathcal{O}_{\mathbb{P}^{n-2}}(-1)$. Since $F \cdot f = -1$ and $F \cdot \widetilde{e}_0 = 1$, we have $F|_G \sim -G_0$. Hence,

$$F \cdot B = F|_G \cdot B = (-G_0 \cdot B)_G.$$

On the other hand, we have

$$F \cdot B = F \cdot \widetilde{e}_0 - F \cdot A = 1 - (-1) = 2.$$

Hence, $(G_0 \cdot B)_G = -2 < 0$ which implies that $B \subset G_0 \subset \widetilde{E}_0$. Thus, $[B] \in \mathbb{R}^+[\widetilde{e}_0]$, a contradiction because $\widetilde{e}_0 \equiv A + B \equiv f + B$. We conclude that all the cases 1, 2, 3 do not happen. Therefore $A \subset \widetilde{E}$. By a similar argument, we also have $B \subset \widetilde{E}$. Now, we take intersection numbers in \widetilde{E} :

$$-1 = (\widetilde{E}_0 \cdot \widetilde{e}_0)_{\widetilde{E}} = (\widetilde{E}_0 \cdot A)_{\widetilde{E}} + (\widetilde{E}_0 \cdot B)_{\widetilde{E}}$$

which implies $\widetilde{E}_0 \cdot A < 0$ or $\widetilde{E}_0 \cdot B < 0$. Hence, $A \subset \widetilde{E}_0$ or $B \subset \widetilde{E}_0$. In both cases we have $[A] \in \mathbb{R}^+[\widetilde{e}_0]$ and $[B] \in \mathbb{R}^+[\widetilde{e}_0]$. It follows that $\mathbb{R}^+[\widetilde{e}_0]$ is an extremal ray in $\overline{\text{NE}}(\widetilde{X})$.

If $n \geq 4$, we have $K_{\widetilde{X}} \cdot \widetilde{e}_0 = 3 - n < 0$. Hence, $\mathbb{R}^+[\widetilde{e}_0]$ is a $K_{\widetilde{X}}$ -negative extremal ray, and we are done by Contraction Theorem.

In case $n = 3$, since we assume $-K_{\widetilde{X}}$ is nef and big, the linear system $|-mK_{\widetilde{X}}|$ defines a morphism for a sufficiently large $m \in \mathbb{N}$ by Base Point Free Theorem. The Stein factorization gives a desired contraction because we have $-K_{\widetilde{X}} \cdot \widetilde{e}_0 = 0$ (note that $\widetilde{E}_0 = \widetilde{e}_0$ for $n = 3$). \square

From now on, we fix the following:

Notation (*)

Assume $n \geq 3$ and put $Y := \mathbb{P}^{n-1} \times \mathbb{P}^1$. Let $p : Y \rightarrow \mathbb{P}^{n-1}$ and $q : Y \rightarrow \mathbb{P}^1$ be the projections. Let C be a fiber of p . Put $H := p^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and $L := q^* \mathcal{O}_{\mathbb{P}^1}(1)$.

Consider $V \in |H + L|$ and $U \in |aH + bL|$ where a and b are non-negative integers. Let S be the complete intersection of U and V . We assume that S is smooth and irreducible. We assume also that C and S intersect transversally at one point and put $y_0 := C \cap S$.

Let h be a fiber of p such that $h \neq C$ and $h \cap S = \emptyset$ and let l be a line in fiber of q such that $l \cap C = \emptyset$ and $l \cap S = \emptyset$.

Let $\pi : X \rightarrow Y$ be the blow-up along C . Put $E := \text{Exc}(\pi)$ and $E_0 := \pi^{-1}(y_0)$. Let e_0 be a line in $E_0 \simeq \mathbb{P}^{n-2}$ and let e be a line in a fiber different from E_0 of the \mathbb{P}^{n-2} -bundle $\pi|_E : E \rightarrow C$. Let H' and L' be the pull backs of H and L by π . Let h' and l' be the strict transforms of h and l by π . Put $S' := \pi_*^{-1} S$.

Let $\beta : \tilde{X} \rightarrow X$ be the blow-up along S' . Put $F := \text{Exc}(\beta)$ and $\tilde{E} := \beta^{-1}(E)$. Let \tilde{H} and \tilde{L} be the pull backs by β of H' and L' . Let f be a fiber of the \mathbb{P}^1 -bundle $\beta|_F : F \rightarrow S'$. Let $\tilde{e}_0, \tilde{e}, \tilde{h}$ and \tilde{l} be the strict transforms by β of e_0, e, h' and l' . Put $V' := \pi_*^{-1} V$ and $\tilde{V} := \beta_*^{-1} V'$.

3 Structure of nef cones

The following is useful to determine the structure of simplicial cones:

Lemma 2. *Let $(D, C) \mapsto D \cdot C$ be a bilinear form of $\mathbb{R}^m \times (\mathbb{R}^m)^*$. Let V be a cone in \mathbb{R}^m and let V^* be its dual cone. Assume that there exist $D_1, D_2, \dots, D_m \in V$ and $C_1, C_2, \dots, C_m \in V^*$ such that $D_i \cdot C_j = \delta_{ij}$ (Kronecker delta). Then, we have*

$$\begin{aligned} V &= \mathbb{R}^+ D_1 + \mathbb{R}^+ D_2 + \dots + \mathbb{R}^+ D_m, \\ V^* &= \mathbb{R}^+ C_1 + \mathbb{R}^+ C_2 + \dots + \mathbb{R}^+ C_m. \end{aligned}$$

Proof. Since D_1, \dots, D_m are linearly independent, for any $D \in V$ there exist real numbers a_1, \dots, a_m such that $D = a_1 D_1 + \dots + a_m D_m$. We have

$$a_i = (a_1 D_1 + \dots + a_m D_m) \cdot C_i = D \cdot C_i \geq 0$$

for $i = 1, \dots, m$. Hence, $D \in \mathbb{R}^+ D_1 + \mathbb{R}^+ D_2 + \dots + \mathbb{R}^+ D_m$. The structure of V^* is similarly determined. \square

Lemma 3. *Let X be a smooth projective variety, V a prime divisor on X and D a divisor on X . If the divisors $D - V$ and $D|_V$ are nef, then D is nef.*

Proof. Let Γ be a curve on X . If $\Gamma \not\subset V$, we have $D \cdot \Gamma = (D - V) \cdot \Gamma + V \cdot \Gamma \geq 0$. If $\Gamma \subset V$, then we have $D \cdot \Gamma = D|_V \cdot \Gamma \geq 0$. \square

Now, we return to our situation (Notation (*) in Section 2).

Proposition 1. *We have*

$$\text{Nef}(\tilde{X}) = \mathbb{R}^+[\tilde{H}] + \mathbb{R}^+[\tilde{L}] + \mathbb{R}^+[\tilde{H} - \tilde{E}] + \mathbb{R}^+[D(a, b)],$$

where

$$D(a, b) := \begin{cases} \tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a = 0 \text{ and } b = 1, \\ 2\tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a = 1 \text{ and } b = 0, \\ 2\tilde{H} + b\tilde{L} - \tilde{E} - F & \text{for } a = 1 \text{ and } b \geq 1, \\ a\tilde{H} + \tilde{L} - \tilde{E} - F & \text{for } a \geq 2 \text{ and } b = 0, \\ a\tilde{H} + b\tilde{L} - \tilde{E} - F & \text{for } a \geq 2 \text{ and } b \geq 1. \end{cases}$$

Proof. We define 1-cycles $l(a)$ and $h(b)$ on \tilde{X} by:

$$l(a) := \begin{cases} \tilde{l} - \tilde{e}_0 - f & (a = 0) \\ \tilde{l} - \tilde{e}_0 - 2f & (a = 1) \\ \tilde{l} - \tilde{e}_0 - af & (a \geq 2) \end{cases}, \quad h(b) := \begin{cases} \tilde{h} - f & (b = 0) \\ \tilde{h} - bf & (b \geq 1). \end{cases}$$

Claim. For any $a \geq 0$, we have $[l(a)] \in \text{NE}(\tilde{X})$.

Proof. Let $t_0 := q(y_0)$ and $t \in \mathbb{P}^1 \setminus \{y_0\}$. Put $y_t := C \cap q^{-1}(t)$. Put also $Y_0 := q^{-1}(t_0)$ and $Y_t := q^{-1}(t)$. We define the curve Γ as follows: If $a = 0$, let Γ be a line in $Y_t \simeq \mathbb{P}^{n-1}$. If $a = 1$, let Γ be a line in Y_t such that $y_t \in \Gamma$ and $S \cap \Gamma \neq \emptyset$. If $a \geq 2$, let Γ be a line in $Y_0 \simeq \mathbb{P}^{n-1}$ such that $y_0 \in \Gamma$ and $\Gamma \subset V$. For any $a \geq 0$, $\Gamma \equiv l$ in Y . Put $\Gamma' := \pi_*^{-1}\Gamma$ and $\tilde{\Gamma} := \beta_*^{-1}\Gamma'$. For $a = 0$ and $a = 1$, we have $\Gamma' + e \equiv l'$. This yields $\tilde{\Gamma} + \tilde{e} \equiv \tilde{l}$ for $a = 0$ (because $\Gamma' \cap S' = \emptyset$) and $\tilde{\Gamma} + \tilde{e} + f \equiv \tilde{l}$ for $a = 1$ (because Γ' and S' intersect transversally at one point). In case $a \geq 2$, we have $\Gamma' + e_0 \equiv l'$ which yields

$$(\tilde{\Gamma} + (a - 1)f) + (\tilde{e}_0 + f) \equiv \tilde{l}$$

because $(S' \cdot \Gamma')_{V'} = a - 1$ and $(S' \cdot e_0)_{V'} = 1$. Thus, for any $a \geq 0$, we have $[l(a)] = [\tilde{\Gamma}] \in \text{NE}(\tilde{X})$. \square

Claim. For any $b \geq 0$, we have $[h(b)] \in \text{NE}(\tilde{X})$.

Proof. We define the curve Δ as follows: If $b = 0$, let Δ be a fiber of $p|_U$ different from C . Note that U is isomorphic to $p(U) \times \mathbb{P}^1$ because $U \sim aH$. If $b \geq 1$, let Δ be a fiber of p such that $\Delta \subset V$ and $\Delta \not\subset S$ (Δ is a fiber of the exceptional divisor of the blow-up $p|_V : V \rightarrow \mathbb{P}^{n-1}$). Since $\Delta \equiv h$ for any $b \geq 0$, we have

$$(S \cdot \Delta)_U = V|_U \cdot \Delta = V \cdot h = 1 \quad \text{for } b = 0,$$

$$(S \cdot \Delta)_V = U|_V \cdot \Delta = U \cdot h = b \quad \text{for } b \geq 1.$$

Put $\tilde{\Delta} := (\pi \circ \beta)_*^{-1} \Delta$. Then, if $b = 0$, we have $\tilde{\Delta} + f \equiv \tilde{h}$ and if $b \geq 1$, $\tilde{\Delta} + bf \equiv \tilde{h}$. Thus, for any $b \geq 0$, we get $[h(b)] = [\tilde{\Delta}] \in \text{NE}(\tilde{X})$. \square

Claim. The divisors $\tilde{H}, \tilde{L}, \tilde{H} - \tilde{E}$ and $D(a, b)$ are all nef.

Proof. We see that $H = p^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and $L = q^* \mathcal{O}_{\mathbb{P}^1}(1)$ are nef. Hence, so are $\tilde{H} = (\pi \circ \beta)^* H$ and $\tilde{L} = (\pi \circ \beta)^* L$. Note that X is isomorphic to $Bl_z(\mathbb{P}^{n-1}) \times \mathbb{P}^1$ where z is the point $p(C) \in \mathbb{P}^{n-1}$. For the blow-up $\varepsilon : Bl_z(\mathbb{P}^{n-1}) \rightarrow \mathbb{P}^{n-1}$ the divisor $\varepsilon^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) - \text{Exc}(\varepsilon)$ is nef. Hence, so is its pull back by the projection $X \rightarrow Bl_z(\mathbb{P}^{n-1})$, which is linearly equivalent to $H' - E$. Therefore, $\tilde{H} - \tilde{E} = \beta^*(H' - E)$ is also nef.

We show that $D(a, b)$ is nef for $(a, b) = (0, 1)$ and for any (a, b) such that $a \geq 1$ and $b \geq 0$.

First, we consider the case $(a, b) = (0, 1)$. Put $H_0 := p^{-1}(p(S))$, $H'_0 := \pi_*^{-1} H_0$ and $\tilde{H}_0 := \beta_*^{-1} H'_0$. Note that we have $S = q^{-1}(q(y_0)) \cap H_0$. Note also that $\pi|_{H'_0} : H'_0 \rightarrow H_0 \simeq \mathbb{P}^{n-2} \times \mathbb{P}^1$ is the blow-up along C and $\beta|_{\tilde{H}_0} : \tilde{H}_0 \rightarrow H'_0$ is an isomorphism. Let L_t be a fiber of q such that $y_0 \notin L_t$. Put $\tilde{L}_t := (\pi \circ \beta)_*^{-1} L_t$. We see that $\tilde{L}_t \cap \tilde{H}_0$ and $F \cap \tilde{H}_0$ are both fibers of the projection

$$(q \circ \pi \circ \beta)|_{\tilde{H}_0} : \tilde{H}_0 \rightarrow \mathbb{P}^1.$$

Hence, we have $\tilde{L}|_{\tilde{H}_0} \sim \tilde{L}_t|_{\tilde{H}_0} \sim F|_{\tilde{H}_0}$. Therefore,

$$(\tilde{H} + \tilde{L} - \tilde{E} - F)|_{\tilde{H}_0} \sim (\tilde{H} - \tilde{E})|_{\tilde{H}_0},$$

which is nef. Since $\widetilde{H}_0 \sim \widetilde{H} - \widetilde{E} - F$, we have

$$(\widetilde{H} + \widetilde{L} - \widetilde{E} - F) - \widetilde{H}_0 \sim \widetilde{L},$$

which is also nef. By Lemma 3, we conclude that $D(0, 1) = \widetilde{H} + \widetilde{L} - \widetilde{E} - F$ is nef on \widetilde{X} .

Now, we show that $D(a, b)$ is nef for $a \geq 1$ and $b \geq 0$. Since $F|_{\widetilde{V}} \in \text{Pic}(\widetilde{V})$ corresponds to $S' \in \text{Pic}(V')$ via the isomorphism $\beta|_{\widetilde{V}} : \widetilde{V} \rightarrow V'$, the divisor $D(a, b)|_{\widetilde{V}}$ is identified with the following:

$$\begin{cases} (2H' + L' - E)|_{V'} - S' & (a = 1, b = 0) \\ (2H' + bL' - E)|_{V'} - S' & (a = 1, b \geq 1) \\ (aH' + L' - E)|_{V'} - S' & (a \geq 2, b = 0) \\ (aH' + bL' - E)|_{V'} - S' & (a \geq 2, b \geq 1). \end{cases}$$

Note that $\pi|_{V'} : V' \rightarrow V$ is the blow-up at the point $y_0 = S \cap C$ and the exceptional divisor is $E \cap V'$. Hence, we have

$$\begin{aligned} S' &\sim (\pi|_{V'})^* S - E|_{V'} \sim (\pi|_{V'})^*(U|_V) - E|_{V'} \sim (aH' + bL')|_{V'} - E|_{V'} \\ &= \begin{cases} (H' - E)|_{V'} & (a = 1, b = 0) \\ (H' + bL' - E)|_{V'} & (a = 1, b \geq 1) \\ (aH' - E)|_{V'} & (a \geq 2, b = 0) \\ (aH' + bL' - E)|_{V'} & (a \geq 2, b \geq 1). \end{cases} \end{aligned}$$

Therefore, $D(a, b)|_{\widetilde{V}}$ corresponds to:

$$\begin{cases} (H' + L')|_{V'} & (a = 1, b = 0) \\ H'|_{V'} & (a = 1, b \geq 1) \\ L'|_{V'} & (a \geq 2, b = 0) \\ 0 & (a \geq 2, b \geq 1), \end{cases}$$

which is nef in any case.

On the other hand, since $\widetilde{V} \sim \widetilde{H} + \widetilde{L} - F$, we have

$$D(a, b) - \widetilde{V} \sim \begin{cases} \widetilde{H} - \widetilde{E} & (a = 1, b = 0) \\ \widetilde{H} + (b - 1)\widetilde{L} - \widetilde{E} & (a = 1, b \geq 1) \\ (a - 2)\widetilde{H} + (\widetilde{H} - \widetilde{E}) & (a \geq 2, b = 0) \\ (a - 2)\widetilde{H} + (b - 1)\widetilde{L} + (\widetilde{H} - \widetilde{E}) & (a \geq 2, b \geq 1). \end{cases}$$

Recall that \tilde{H}, \tilde{L} and $\tilde{H} - \tilde{E}$ are nef. Hence, so is $D(a, b) - \tilde{V}$. By Lemma 3, we conclude that $D(a, b)$ is nef. \square

We have the following table of intersection numbers.

	\tilde{H}	\tilde{L}	\tilde{E}	F
\tilde{l}	1	0	0	0
\tilde{h}	0	1	0	0
\tilde{e}_0	0	0	-1	1
f	0	0	0	-1

By definition of $l(a)$, $h(b)$ and $D(a, b)$, for $(a, b) = (0, 1)$ and for any (a, b) such that $a \geq 1$ and $b \geq 0$, we have:

	\tilde{H}	\tilde{L}	$\tilde{H} - \tilde{E}$	$D(a, b)$
$l(a)$	1	0	0	0
$h(b)$	0	1	0	0
\tilde{e}_0	0	0	1	0
f	0	0	0	1

Now, the proposition follows from Lemma 2 because we have $\rho(\tilde{X}) = 4$. \square

Remark. In the proof, we have also shown that

$$\overline{\text{NE}}(\tilde{X}) = \mathbb{R}^+[l(a)] + \mathbb{R}^+[h(b)] + \mathbb{R}^+[\tilde{e}_0] + \mathbb{R}^+[f].$$

4 Self intersection numbers of anti-canonical divisors

The purpose of this section is to prove the following:

Proposition 2. *If $a = 1$, we have*

$$(-K_{\tilde{X}})^n = \frac{(7-b)n}{2}(n-1)^{n-1} - 2(n-1)(n-2)^{n-1} + (n-3)^n.$$

If $a \neq 1$, we have

$$(-K_{\tilde{X}})^n = (n-a)^{n-1} \frac{(-3a+2+ab)n+a^2-ab}{(a-1)^2} + (n-1)^{n-1} \frac{(a^2-b)n-a+b}{(a-1)^2} - 2(n-1)(n-2)^{n-1} + (n-3)^n.$$

We prepare some lemmas.

Lemma 4. *Let D be a divisor on a smooth projective variety Y of dimension $n \geq 3$ and let S be a smooth subvariety in Y of codimension $r \geq 2$. Let $\mu : Z \rightarrow Y$ be the blow-up along S . Let F be the exceptional divisor of μ . Then, for $k = 1, 2, \dots, n$, we have*

$$(\mu^* D)^{n-k} F^k = (-1)^{r-1} (D|_S)^{n-k} s_{k-r}(N_{S/Y}^*)$$

where $s_{k-r}(N_{S/Y}^*)$ denotes the Segre classes of the conormal bundle $N_{S/Y}^*$.

Proof. We follow the notation in [4] Chapter 3 and Appendix B, i.e. for a vector space V , the projectivization $\mathbb{P}(V)$ denotes the set of lines in V . Consider the \mathbb{P}^{r-1} -bundle $\mu|_F : F = \mathbb{P}(N_{S/Y}) \rightarrow S$. Let $\mathcal{O}(1)$ be the dual bundle of the tautological line bundle $\mathcal{O}(-1)$ associated to $N_{S/Y}$. By a definition of Segre classes, we have

$$\begin{aligned} ((\mu|_F)^*(D|_S))^{n-k} \mathcal{O}(1)^{k-1} &= ((\mu|_F)^*(D|_S))^{n-k} \mathcal{O}(1)^{(r-1)+(k-r)} \\ &= (D|_S)^{n-k} s_{k-r}(N_{S/Y}). \end{aligned}$$

This yields

$$\begin{aligned} (\mu^* D)^{n-k} F^k &= (\mu^* D|_F)^{n-k} (F|_F)^{k-1} \\ &= ((\mu|_F)^*(D|_S))^{n-k} \mathcal{O}(-1)^{k-1} \\ &= (-1)^{k-1} (D|_S)^{n-k} s_{k-r}(N_{S/Y}) \\ &= (-1)^{k-1} (D|_S)^{n-k} (-1)^{k-r} s_{k-r}(N_{S/Y}^*) \\ &= (-1)^{r-1} (D|_S)^{n-k} s_{k-r}(N_{S/Y}^*). \end{aligned}$$

□

Now, we return to our situation (Notations (*) in Section 2). However, in what follows, we put $h := H|_S$ and $l := L|_S$.

Lemma 5. *We have*

$$h^{n-2} = a + b, \quad h^{n-3}l = a, \quad l^2 \equiv 0.$$

Proof. Note that $S = UV \equiv (aH + bL)(H + L)$. Since $H^n = 0$, $H^{n-1}L = 1$ and $L^2 \equiv 0$, we obtain

$$\begin{aligned} h^{n-2} &= H^{n-2}S = aH^n + (a+b)H^{n-1}L = a+b, \\ h^{n-3}l &= H^{n-3}LS = aH^{n-1}L + (a+b)H^{n-2}L^2 = a, \\ l^2 &= L^2S \equiv 0. \end{aligned}$$

□

For $a \geq 1$, we put $P(m) := \sum_{i=0}^m a^i$ and $Q(m) := \sum_{i=0}^m (ia^{i-1}b + (m-i)a^i)$.

Lemma 6. *For $m = 1, 2, \dots, n-2$, the m -th Segre classe is given by*

$$s_m(N_{S/Y}^*) = P(m)h^m + Q(m)h^{m-1}l.$$

Proof. Put $u := U|_S$ and $v := V|_S$. Since $S = U \cap V$ is a complete intersection, we have

$$N_{S/Y} = N_{U/Y}|_S \oplus N_{V/Y}|_S = U|_S \oplus Y|_S = u \oplus v.$$

Hence, $N_{S/Y}^* = (-u) \oplus (-v)$. By Whitney formula, we obtain

$$c(N_{S/Y}^*) = (1-u)(1-v).$$

By the equality $c \cdot s = 1$ between the total Chern classe and the total Segre classe, we get

$$s(N_{S/Y}^*) = \frac{1}{1-u} \cdot \frac{1}{1-v} = (1+u+u^2+\dots) \cdot (1+v+v^2+\dots),$$

whose homogeneous part of degree m equals $\sum_{i+j=m} u^i v^j$. Since $l^2 \equiv 0$, we have

$$u^i v^j = (ah + bl)^i (h + l)^j = a^i h^{i+j} + (ia^{i-1}b + ja^i)h^{i+j-1}l.$$

Therefore,

$$\begin{aligned} s_m(N_{S/Y}^*) &= \sum_{i+j=m} u^i v^j \\ &= \sum_{i+j=m} (a^i h^{i+j} + (ia^{i-1}b + ja^i)h^{i+j-1}l) \\ &= \left(\sum_{i+j=m} a^i \right) h^m + \left(\sum_{i+j=m} (ia^{i-1}b + ja^i) \right) h^{m-1}l. \end{aligned}$$

□

Put

$$\begin{aligned}
I_n &:= \sum_{k=2}^n \binom{n}{k} (-1)^k P(k-2) n^{n-k}, \\
I'_n &:= \sum_{k=2}^n \binom{n}{k} (-1)^k k P(k-2) n^{n-k}, \\
J_n &:= \sum_{k=2}^n \binom{n}{k} (-1)^k Q(k-2) n^{n-k}.
\end{aligned}$$

Lemma 7. *If $a = 1$, we have*

$$\begin{aligned}
I_n &= n^n - (2n-1)(n-1)^{n-1}, \\
I'_n &= n(n-1)^{n-1}, \\
J_n &= \frac{b+1}{2} ((5n-2)(n-1)^{n-1} - 2n^n).
\end{aligned}$$

If $a \geq 2$, we have

$$\begin{aligned}
I_n &= \frac{(n-a)^n + (a-1)n^n - a(n-1)^n - a(n-1)^n}{a(a-1)}, \\
I'_n &= \frac{n}{a-1} ((n-1)^{n-1} - (n-a)^{n-1}), \\
J_n &= \frac{(a+b-2ab)(n-a) - ab(a-1)n}{a^2(a-1)^2} (n-a)^{n-1} \\
&\quad + \frac{(a-1)n + (a+b-2)(n-1)}{(a-1)^2} (n-1)^{n-1} - \frac{a+b}{a^2} n^n.
\end{aligned}$$

Proof. For $a = 1$, we have

$$P(k-2) = k-1, \quad Q(k-2) = \frac{b+1}{2} (k^2 - 3k + 2).$$

If $a \geq 2$, we put $\theta := 1/(a^2 - a)$. Then, we have

$$\begin{aligned}
P(k-2) &= \theta(a^k - a), \\
Q(k-2) &= \theta^2((a+b-2ab)a^k + b(a-1)ka^k - a^2(a-1)k + a^2(a+b-2)).
\end{aligned}$$

The statement is verified by direct computations using the following equalities for $x = a$ and $x = 1$:

$$\begin{aligned}\sum_{k=2}^n \binom{n}{k} (-x)^k n^{n-k} &= (n-x)^n + (x-1)n^n, \\ \sum_{k=2}^n \binom{n}{k} k (-x)^k n^{n-k} &= xn^n - xn(n-x)^{n-1}, \\ \sum_{k=2}^n \binom{n}{k} k^2 (-x)^k n^{n-k} &= x(x-1)n^2(n-x)^{n-2} + xn^n.\end{aligned}$$

□

Proof of Proposition 2. First, we consider the case $(a, b) = (0, 1)$. Put $L_0 := q^{-1}(q(y_0))$ and $H_0 := p^{-1}(p(S))$. Note that S is a hyperplane in $L_0 \simeq \mathbb{P}^{n-1}$. Let L'_0 and H'_0 be the strict transforms by π of L_0 and H_0 . Then S' is the complete intersection of L'_0 and H'_0 . Since $H'_0 \sim H' - E$, $L'_0 \sim L'$ and $L'|_{S'} \sim 0$, we have $N_{S'/X} \simeq \mathcal{O}_{S'}(H' - E) \oplus \mathcal{O}_{S'}$. As in the proof of Lemma 6, this yields

$$s_m(N_{S'/X}^*) = (H'|_{S'} - E|_{S'})^m \text{ for } m = 1, 2, \dots, n-2.$$

On the other hand, we have

$$-K_X|_{S'} \sim (nH' + 2L' - (n-2)E)|_{S'} \sim nH'|_{S'} - (n-2)E|_{S'}.$$

We observe that $\pi|_{S'} : S' \rightarrow S \simeq \mathbb{P}^{n-2}$ is the blow-up at y_0 . We have $H'|_{S'} \sim (\pi|_{S'})^* \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ and $\text{Exc}(\pi|_{S'}) = E|_{S'}$. Note that

$$(H'|_{S'})(E|_{S'}) \equiv 0, \quad (H'|_{S'})^{n-2} = 1 \text{ and } (E|_{S'})^{n-2} = (-1)^{n-3}.$$

By Lemma 4 for $r = 2$, we obtain $\beta^*(-K_X)^{n-1}F = 0$ and for $k = 2, \dots, n$,

$$\begin{aligned}\beta^*(-K_X)^{n-k}F^k &= -(-K_X|_{S'})^{n-k}s_{k-2}(N_{S'/X}^*) \\ &= -(nH'|_{S'} - (n-2)E|_{S'})^{n-k}(H'|_{S'} - E|_{S'})^{k-2} \\ &= (n-2)^{n-k} - n^{n-k}.\end{aligned}$$

Since X is isomorphic to $\mathbb{P}^1 \times Bl_z(\mathbb{P}^{n-1})$ where z is a point in \mathbb{P}^{n-1} , we have

$$(-K_X)^n = 2n(n^{n-1} - (n-2)^{n-1}).$$

It follows that

$$\begin{aligned}
(-K_{\tilde{X}})^n &= (\beta^*(-K_X) - F)^n \\
&= (-K_X)^n + \sum_{k=2}^n \binom{n}{k} (-1)^k \beta^*(-K_X)^{n-k} F^k \\
&= 2n(n^{n-1} - (n-2)^{n-1}) + \sum_{k=2}^n \binom{n}{k} (-1)^k ((n-2)^{n-k} - n^{n-k}) \\
&= 2n^n - (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n.
\end{aligned}$$

In case $(a, b) \neq (0, 1)$, since we cannot necessarily describe $S' \subset X$ as a complete intersection (remark that $U' \cap V' = S' \cup E_0$ where U' and V' are the strict transforms by π of U and V), it seems hard to compute $(-K_{\tilde{X}})^n$ directly from $(-K_X)^n$. We avoid this difficulty by considering a flip of \tilde{X} :

Step 1. Let $\mu : Z \rightarrow Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$ be the blow-up along $S = U \cap V$. Let F_Z be the exceptional divisor of μ . We have

$$\mu^*(-K_Y)^n = (-K_Y)^n = (nH + 2L)^n = 2n^n.$$

By Lemma 4 for $r = 2$, we have $\mu^*(-K_Y)^{n-1} F_Z = 0$ and

$$\mu^*(-K_Y)^{n-k} F_Z^k = -(-K_Y|_S)^{n-k} s_{k-2}(N_{S/Y}^*) \quad \text{for } k = 2, \dots, n.$$

Therefore,

$$\begin{aligned}
(-K_Z)^n &= (\mu^*(-K_Y) - F_Z)^n \\
&= \mu^*(-K_Y)^n + \sum_{k=1}^n \binom{n}{k} \mu^*(-K_Y)^{n-k} (-F_Z)^k \\
&= 2n^n - \sum_{k=2}^n \binom{n}{k} (-1)^k (-K_Y|_S)^{n-k} s_{k-2}(N_{S/Y}^*).
\end{aligned}$$

Here, we have $-K_Y|_S \sim (nH + 2L)|_S = nh + 2l$. By Lemma 5 and 6, $(-K_Y|_S)^{n-k} s_{k-2}(N_{S/Y}^*)$ is equal to

$$(3a + b)P(k-2)n^{n-k} - 2aP(k-2)kn^{n-k-1} + aQ(k-2)n^{n-k}.$$

Thus,

$$(-K_Z)^n = 2n^n - (3a + b)I_n + \frac{2a}{n}I'_n - aJ_n.$$

Now, substituting the result of Lemma 7, we conclude that if $a = 1$, we have

$$(-K_Z)^n = \frac{(7-b)n}{2}(n-1)^{n-1},$$

if $a \geq 2$, we have

$$(-K_Z)^n = (n-a)^{n-1} \frac{(-3a+2+ab)n+a^2-ab}{(a-1)^2} + (n-1)^{n-1} \frac{(a^2-b)n-a+b}{(a-1)^2}.$$

Step 2. Let $\alpha : \tilde{Z} \rightarrow Z$ be the blow-up along the curve $C' := \mu_*^{-1}C$ with the exceptional divisor G . Note that $N_{C'/Z} = \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-2)} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence, we have $-K_Z \cdot C' = 1$ and $\deg(N_{C'/Z}^*) = 1$. Using Lemma 4 for $r = n-1$, we have the following:

$$\begin{aligned} \alpha^*(-K_Z)^{n-k} G^k &= 0 \quad \text{for } k = 1, 2, \dots, n-2, \\ \alpha^*(-K_Z) G^{n-1} &= (-1)^n (-K_Z \cdot C') = (-1)^n, \\ G^n &= (-1)^n s_1(N_{C'/Z}^*) = (-1)^{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} (-K_{\tilde{Z}})^n &= (\alpha^*(-K_Z) - (n-2)G)^n \\ &= (\alpha^*(-K_Z))^n + (-1)^{n-1} n(n-2)^{n-1} \alpha^*(-K_Z) G^{n-1} + (-1)^n (n-2)^n G^n \\ &= (-K_Z)^n - n(n-2)^{n-1} - (n-2)^n \\ &= (-K_Z)^n - 2(n-1)(n-2)^{n-1}. \end{aligned}$$

Step 3. We observe that \tilde{X} and \tilde{Z} are connected by a flip. We have

$$(-K_{\tilde{X}})^n = (-K_{\tilde{Z}})^n + (n-3)^n.$$

To see this, put $\Gamma_0 := \mu^{-1}(y_0)$ and $\widetilde{\Gamma_0} := \alpha_*^{-1}\Gamma_0$. Let $\gamma : W \rightarrow \tilde{Z}$ be the blow-up along the curve $\widetilde{\Gamma_0}$ and let M be the exceptional divisor of γ . Note that $M \simeq \mathbb{P}^{n-2} \times \mathbb{P}^1$ and $N_{M/W} \simeq \mathcal{O}_{\mathbb{P}^{n-2} \times \mathbb{P}^1}(-1, -1)$. The contraction map sending M to \mathbb{P}^{n-2} is nothing but the blow-up $\delta : W \rightarrow \tilde{X}$ along $\widetilde{E_0} \simeq \mathbb{P}^{n-2}$.

We have $K_W \sim \delta^* K_{\tilde{X}} + M$ and $K_W \sim \gamma^* K_{\tilde{Z}} + (n-2)M$. Hence,

$$\delta^*(-K_{\tilde{X}}) \sim \gamma^*(-K_{\tilde{Z}}) - (n-3)M.$$

Note that $N_{\widetilde{\Gamma_0}/\tilde{Z}} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{\oplus(n-1)}$. Hence, we have $-K_{\tilde{Z}} \cdot \widetilde{\Gamma_0} = 3-n$ and $\deg(N_{\widetilde{\Gamma_0}/\tilde{Z}}^*) = n-1$. As in Step 2, we obtain $\gamma^*(-K_{\tilde{Z}}) M^{n-1} = (-1)^{n+1}(n-3)$, $M^n = (-1)^{n+1}(n-1)$ and $\gamma^*(-K_{\tilde{Z}})^{n-k} M^k = 0$ for $k = 1, \dots, n-2$.

Thus,

$$\begin{aligned}
(-K_{\tilde{X}})^n &= (\delta^*(-K_{\tilde{X}}))^n \\
&= (\gamma^*(-K_{\tilde{Z}}) - (n-3)M)^n \\
&= (\gamma^*(-K_{\tilde{Z}}))^n + (-1)^{n-1}n(n-3)^{n-1}\gamma^*(-K_{\tilde{Z}})M^{n-1} + (-1)^n(n-3)^nM^n \\
&= (-K_{\tilde{Z}})^n + (n-3)^n.
\end{aligned}$$

By Step 2 and Step 3, we obtain

$$(-K_{\tilde{X}}) = (-K_Z)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n.$$

Substituting the result of Step 1, we complete the proof of Proposition 2. \square

Proposition 3. *For $(a, b) = (0, 1)$ and for (a, b) such that $1 \leq a \leq 3$ and $0 \leq b \leq 3$, we have $(-K_{\tilde{X}})^n > 0$ for any $n \geq 3$.*

Proof. For each case, we compute $(-K_{\tilde{X}})^n$ using Proposition 2. Note that we have $(n-1)^n - 2(n-1)(n-2)^{n-1} > 0$ for any $n \geq 3$.

If $(a, b) = (0, 1)$, then

$$\begin{aligned}
(-K_{\tilde{X}})^n &= 2n^n - (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&\geq 2(n-1)^n - (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&\geq (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&> 0.
\end{aligned}$$

Assume $b \leq 3$. If $a = 1$,

$$\begin{aligned}
(-K_{\tilde{X}})^n &= \frac{7-b}{2}n(n-1)^{n-1} - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&\geq 2n(n-1)^{n-1} - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&> (n-1)^n - 2(n-1)(n-2)^{n-1} + (n-3)^n \\
&> 0.
\end{aligned}$$

If $a = 2$,

$$\begin{aligned}
(-K_{\tilde{X}})^n &= (4n-2)(n-1)^{n-1} - 6(n-1)(n-2)^{n-1} \\
&\quad - b((n-1)^n - 2(n-1)(n-2)^{n-1}) + (n-3)^n \\
&\geq (4n-2)(n-1)^{n-1} - 6(n-1)(n-2)^{n-1} \\
&\quad - 3((n-1)^n - 2(n-1)(n-2)^{n-1}) + (n-3)^n \\
&= (n-1)^n + (n-3)^n \\
&> 0.
\end{aligned}$$

If $a = 3$,

$$\begin{aligned}
4(-K_{\tilde{X}})^n &= -3(n+1)(n-3)^{n-1} - 8(n-1)(n-2)^{n-1} + (9n-3)(n-1)^{n-1} \\
&\quad - b((n-1)^n - 3(n-1)(n-3)^{n-1}) \\
&\geq -3(n+1)(n-3)^{n-1} - 8(n-1)(n-2)^{n-1} + (9n-3)(n-1)^{n-1} \\
&\quad - 3((n-1)^n - 3(n-1)(n-3)^{n-1}) \\
&= (6n-12)(n-3)^{n-1} + 6n(n-1)^{n-1} - 8(n-1)(n-2)^{n-1} \\
&> 6n(n-1)^{n-1} - 8(n-1)(n-2)^{n-1} \\
&> 4((n-1)^{n-1} - 2(n-1)(n-2)^{n-1}) \\
&> 0.
\end{aligned}$$

□

Remark. More precise estimations show that we have $(-K_{\tilde{X}})^n > 0$ for any $n \geq 3$ in the cases: $a = 1$ and $b \leq 5$; $a = 2$ and $b \leq 6$; $a = 3$ and $b \leq 8$. In case $a \geq 4$, the positivity of $(-K_{\tilde{X}})^n$ is independent of the value b . For example, if $a = 15$ we have $(-K_{\tilde{X}})^4 = -306b - 285 < 0$ and $(-K_{\tilde{X}})^5 = 3056b + 1344 > 0$ for any $b \in \mathbb{N}$.

5 Proof of Theorem 1

By the canonical bundle formula for the blow-ups π and β , we have

$$K_{\tilde{X}} \sim \beta^* K_X + F \sim \beta^*(\pi^* K_Y) + (n-2)\tilde{E} + F.$$

Combining with $-K_Y \sim nH + 2L$, we get

$$-K_{\tilde{X}} \sim n\tilde{H} + 2\tilde{L} - (n-2)\tilde{E} - F.$$

First, we consider the case $a \geq 2$ and $b \geq 1$. We rewrite $-K_{\tilde{X}}$ by means of the generators of $Nef(\tilde{X})$:

$$-K_{\tilde{X}} = (3-a)\tilde{H} + (2-b)\tilde{L} + (n-3)(\tilde{H} - \tilde{E}) + (a\tilde{H} + b\tilde{L} - \tilde{E} - F).$$

By Proposition 1, we see that $-K_{\tilde{X}}$ is nef if and only if $3-a \geq 0$, $2-b \geq 0$ and $n-3 \geq 0$. Since the numerical equivalence classes of ample divisors are interior points of the nef cone ([7] Theorem 1.4.23), it follows that $-K_{\tilde{X}}$ is ample if and only if $3-a > 0$, $2-b > 0$ and $n-3 > 0$.

In the other cases, we argue similarly in the following forms:

- For $a = 0$ and $b = 1$,

$$-K_{\tilde{X}} \sim 2\tilde{H} + \tilde{L} + (n-3)(\tilde{H} - \tilde{E}) + (\tilde{H} + \tilde{L} - \tilde{E} - F).$$

- For $a = 1$ and $b = 0$,

$$-K_{\tilde{X}} \sim \tilde{H} + \tilde{L} + (n-3)(\tilde{H} - \tilde{E}) + (2\tilde{H} + \tilde{L} - \tilde{E} - F).$$

- For $a = 1$ and $b \geq 1$,

$$-K_{\tilde{X}} \sim \tilde{H} + (2-b)\tilde{L} + (n-3)(\tilde{H} - \tilde{E}) + (2\tilde{H} + b\tilde{L} - \tilde{E} - F).$$

- For $a \geq 2$ and $b = 0$,

$$-K_{\tilde{X}} \sim (3-a)\tilde{H} + \tilde{L} + (n-3)(\tilde{H} - \tilde{E}) + (a\tilde{H} + \tilde{L} - \tilde{E} - F).$$

Finally, we conclude that $-K_{\tilde{X}}$ is nef if and only if $n \geq 3$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1) \text{ or } (3, 2).$$

Moreover, $-K_{\tilde{X}}$ is ample if and only if $n \geq 4$ and

$$(a, b) = (0, 1), (1, 0), (1, 1), (2, 0) \text{ or } (2, 1).$$

In general, a nef divisor D is big if and only if $D^n > 0$ ([7] Theorem 2.2.16). Thus, the proof of Theorem is completed by Proposition 3. \square

6 Other examples

Details in this section can be verified by the methods in Sections 3 and 4. We keep all the notations (*) in Section 2 except that the divisors H and L are replaced by appropriate ones.

In the following examples 1, 2 and 3, we put $Y = \mathbb{P}^n$ with $n \geq 4$.

Example 1. Let C be a line and S an $(n-2)$ -plane. Assume $C \cap S \neq \emptyset$. We consider $H := \mathcal{O}_{\mathbb{P}^n}(1)$ and $\tilde{H} := (\pi \circ \beta)^*H$. Then, we have

$$\text{Nef}(\tilde{X}) = \mathbb{R}^+[\tilde{H}] + \mathbb{R}^+[\tilde{H} - \tilde{E}] + \mathbb{R}^+[2\tilde{H} - \tilde{E} - F],$$

$$-K_{\tilde{X}} \sim (n+1)\tilde{H} - (n-2)\tilde{E} - F = 2\tilde{H} + (n-3)(\tilde{H} - \tilde{E}) + (2\tilde{H} - \tilde{E} - F).$$

Hence, \tilde{X} is a Fano manifold for any $n \geq 4$.

Example 2. Let C be a line. Let S be the complete intersection of a hyperplane and a hyperquadric. Assume that C and S intersect transversally at one point. Then \tilde{X} is a Fano manifold for any $n \geq 4$. Indeed, the structure of nef cone and the description of the anti-canonical divisor for \tilde{X} are completely same as in Example 1. Even if the intersection $C \cap S$ consists of *two* points, \tilde{X} remains Fano, while the exceptional locus of the small contraction has two irreducible components.

Example 3. Let $P \subset Y = \mathbb{P}^n$ be a 2-plane and C a smooth conic on $P \simeq \mathbb{P}^2$. Let S be an $(n-2)$ -plane such that $\sharp(C \cap S) = 2$. Then, we have

$$\text{Nef}(\tilde{X}) = \mathbb{R}^+[\tilde{H}] + \mathbb{R}^+[2\tilde{H} - \tilde{E}] + \mathbb{R}^+[3\tilde{H} - \tilde{E} - F],$$

$$-K_{\tilde{X}} \sim (n+1)\tilde{H} - (n-2)\tilde{E} - F = (4-n)\tilde{H} + (n-3)(2\tilde{H} - \tilde{E}) + (3\tilde{H} - \tilde{E} - F).$$

We see that $-K_{\tilde{X}}$ is nef only for $n = 4$. Moreover, we have $(-K_{\tilde{X}})^4 = 353 > 0$. Hence \tilde{X} is a weak Fano manifold for $n = 4$.

Example 4. Put $Y := \mathbb{P}^2 \times \mathbb{P}^2$. Let C be a line in a fiber of a projection $Y \rightarrow \mathbb{P}^2$ and S a fiber of the other projection such that $C \cap S \neq \emptyset$. Then \tilde{X} is a Fano 4-fold. Indeed, we are able to show:

Proposition 4. *Let $Y := \mathbb{P}^{n-2} \times \mathbb{P}^2$ with $n \geq 3$. Let C be a smooth plane curve of degree d in a fiber of the projection $p : Y \rightarrow \mathbb{P}^{n-2}$. Let S be a fiber of the projection $q : Y \rightarrow \mathbb{P}^2$ such that $C \cap S \neq \emptyset$. Then \tilde{X} is a weak Fano manifold if and only if*

$$(n, d) = (3, 1), (3, 2), (3, 3), (4, 1) \text{ or } (5, 1).$$

Moreover, \tilde{X} is a Fano manifold if and only if $(n, d) = (4, 1)$.

Proof. Put $H := p^* \mathcal{O}_{\mathbb{P}^{n-2}}(1)$ and $L := q^* \mathcal{O}_{\mathbb{P}^2}(1)$. Then we have

$$\text{Nef}(\tilde{X}) = \mathbb{R}^+[\tilde{H}] + \mathbb{R}^+[\tilde{L}] + \mathbb{R}^+[\tilde{H} + d\tilde{L} - \tilde{E}] + \mathbb{R}^+[\tilde{H} + d\tilde{L} - \tilde{E} - F].$$

Since $K_{\tilde{X}} \sim (\pi \circ \beta)^* K_Y + (n-2)\tilde{E} + F$ and $-K_Y \sim (n-1)H + 3L$, we have

$$\begin{aligned} -K_{\tilde{X}} &\sim (n-1)\tilde{H} + 3\tilde{L} - (n-2)\tilde{E} - F \\ &= \tilde{H} + (3-d(n-2))\tilde{L} + (n-3)(\tilde{H} + d\tilde{L} - \tilde{E}) + (\tilde{H} + d\tilde{L} - \tilde{E} - F). \end{aligned}$$

Hence, $-K_{\tilde{X}}$ is nef (resp. ample) if and only if $3-d(n-2)$ and $n-3$ are positive (resp. strictly positive). On the other hand, we obtain

$$(-K_{\tilde{X}})^n = 4n(n-1)^{n-1} + (n-2)^{n-1}(d(d-3)n - 2d^2 + 2) + (n-3)^n,$$

which is strictly positive for $(n, d) = (3, 1), (3, 2), (3, 3), (4, 1)$ and $(5, 1)$. \square

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